without reducing th eir strength properties. Commercial trials of molds for die casting made by the proposed method showed that the resistance of the tool to crazing is 20% greater than for dies made by the traditional method of bulk quenching.

NOTATION

n, exponent of the approximating parabola; T, temperature of the body in an arbitrary plane; T_s, specimen surface temperature; T_i, initial temperature; T_{cr}, melting point of the heated material; T_a, ambient temperature; y, distance from the symmetry axis of the parabola to the plane with arbitrary temperature; X, thickness of the layer heated to the moment of time t; t, arbitrary time; t_1, time to beginning of melting of surface; ξ, depth of liquid zone; t_2, time to melting of specimen to a specified depth; $dQ_{e\ell}$, quantity of heat transmitted by the electron beam; dQ_{ac}, quantity of accumulated heat; Q_{rad}, quantity of heat lost by radiation; Q_{cr}, quantity of heat expended on melting; $q_{e\ell}$, specific power of the electron-beam heating; L, thickness of the plane body; ρ, density of the heated material; r, heat of fusion of the material; C, radiation coefficient; λ, thermal conductivity; $H_\square$, microhardness; K_D, degree of dendritic segregation; F, heating area; h, distance from the surface; x, linear size of the scanned section.

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GROUP ANALYSIS OF THE HEAT-CONDUCTION EQUATION IN DISPLACEMENTS OF ISOTHERMAL SURFACES. 2. OBTAINING INVARIANT SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

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UDC 536.24.01

A rule is formulated and examples are given for constructing particular solutions of boundary-value problems of heat conduction.

The present paper is a development of the results [1] referring to obtaining invariant solutions for the equation of the nonstationary heat-conduction process

$$x'_\tau = f(T)\, x''_{TT}\, (x'_T)^{-2} - f'(T)\, (x'_T)^{-1}, \tag{1}$$

written for the location x of isothermal surfaces T = idem at the time τ. The function f(T) in (1) is related to the temperature dependences of the heat-conduction coefficient $\lambda(\tilde{T})$ and

Sergo Ordzhonikidze Ural Aviation Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 54, No. 1, pp. 129-133, January, 1988. Original article submitted October 14, 1986.

 0022-0841/88/5401-0100 $12.50

the volume specific heat $c\rho(\tilde{T})$ as follows:

$$f(T) = f(F(\tilde{T})) = \lambda(\tilde{T})/c\rho(\tilde{T}), \tag{2}$$

where the model temperature T is defined as

$$T = \int c\rho(\tilde{T}')\, d\tilde{T}'.$$

The idea about how to formulate the initial and boundary conditions corresponding to boundary-value problems of nonstationary heat conduction in both domains with a fixed boundary and for a given law of its motion in time or for finding this law from an additional condition for the substance phase transition (from the Stefan condition) in the terminology of the invariant solutions obtained in [1] arises. Obtaining such particular solutions is of indubitable interest for applications and is the subject of the present paper.

Firstly, let us note that in order to go from the invariant solutions of (1) obtained in [1] to the solution of boundary-value problems for this equation, the following question must be resolved: what conditions should the manifold satisfy so that the solution of the boundary-value problem would be invariant relative to the allowable transformation group.

This latter requirement is evidently assured if the selection rule for the initial and boundary conditions is followed in conformity with the structure of the invariant solution. Let us show in what deductions the application of this rule results when applied to the invariant solutions of (1) established in [1], the major portion of which can be given the form

$$x(T, \tau) = w(T, \tau)\, v(\mu), \tag{3}$$

where $w(T, \tau)$, $\mu = \mu(T, \tau)$, $v(\mu)$ are known functions.

The initial condition for (1) takes the form

$$x_0(T) = w(T, 0)\, v(\mu(T, 0)), \tag{4}$$

from which there follows that the initial distribution $x_0(T)$ can be selected only by means of (4) on the basis of a preliminarily found invariant solution of (1).

We formulate the boundary conditions for the most complex case when at least one boundary of the body moves in time.

1. The boundary moves according to the law $T = a(\tau)$. then the boundary conditions for (1) is such

$$a(\tau) = w(\psi(\tau), \tau)\, v(\psi(\tau), \tau), \tag{5}$$

so that if $a(\tau)$ is given, then $\psi(\tau)$ is to be determined from (5). If $\psi(\tau)$ is given then $a(\tau)$ is determined according to the algorithm mentioned.

2. The boundary moves according to the law $x = a(\tau)$ and the gradient $x_T' = \phi(T)g(\tau)$ is known on it ($\phi(T)$ is the function for going over to x_T' from the formula $\lambda(\tilde{T})/x'_{\tilde{T}}$ for the heat-flux density in physical variables). By using the invariant solution (3), we set up the condition on $g(\tau)$ in the form of a system of equations

$$\varphi(T)\, g(\tau) = w_T'(T, \tau)\, v(\mu) + w(T, \tau)\, v_\mu'(\mu)\, \mu_T'(T, \tau), \quad a(\tau) = w(T, \tau)\, v(\mu), \tag{6}$$

by eliminating T we obtain the relation between the functions $g(\tau)$ and $a(\tau)$, one of which can be given arbitrarily.

3. Conditions at infinity as $x \to \infty$ for $T \to T_0$ or $x \to \infty$ as $x_T' \to \infty$ are given on a half-line (on a semiinfinite interval). The condition

$$w(T, \tau)\, v(\mu)|_{T\to T_0} \to \infty \quad \text{(uniformly in } \tau\text{)}$$

or

$$[w_T'(T, \tau)\, v(\mu) + w(T, \tau)\, v_\mu'(\mu)\, \mu_T'(T, \tau)]|_{T\to T_*} \to \infty, \tag{7}$$

should be satisfied for the invariant solution (3), where T_* is determined from the relationship

$$w(T, \tau)\, v(\mu)|_{T\to T_*} \to \infty. \tag{8}$$

4. The condition for ideal heat insulation of the moving boundary takes the form $x = a(\tau)$ as $x_T' \to \infty$ so that by using the form (3) of the invariant solution we obtain a system of equations

$$a(\tau) = w(T_*, \tau)\, v(\mu(T_*, \tau)),$$
$$[w'_T(T, \tau)\, v(\mu) + w(T, \tau)\, v'_\mu(\mu)\, \mu'_T(T, \tau)]|_{T \to T_*} \to \infty. \tag{9}$$

5. The boundary moves according to the law $x = a(\tau)$, and boundary conditions of the third kind are given on it

$$\alpha(\tau)\, x'_T [F(T) - F(T_c)] = \varphi(T), \tag{10}$$

where $\alpha(\tau)$ is the convective heat elimination factor, and $F(T)$ and $F(T_c)$ are known functions of the model temperature T on the moving boundary and in the environment associated with the passage from $\tilde{T}$ to T.

We obtain the following system of equations interconnecting $a(\tau)$, $\alpha(\tau)$ and T_c after T has been eliminated, for the invariant solution in the form (3)

$$a(\tau) = w(T, \tau)\, v(\mu),$$
$$\alpha(\tau)\, [w'_T(T, \tau)\, v(\mu) + w(T, \tau)\, v'_\mu(\mu)\, \mu'_T(T, \tau)]\, [F(T) - F(T_c)] = \varphi(T). \tag{11}$$

6. The condition

$$[x'_\tau x'_T = \varphi(T)/L]|_{T=T_p}, \tag{12}$$

is satisfied for the solution of the single-phase Stefan problem on a moving interphasal boundary $x = s(\tau)$, where L is the heat of the phase transition referred to unit volume, and T_p is the model temperature of the phase transition.

For the invariant solution

$$s(\tau) = w(T_p, \tau)\, v(\mu(T_p, \tau)) \tag{13}$$

the location of the boundary becomes known if the following equality is satisfied

$$\{[w'_\tau(T, \tau)\, v(\mu) + w(T, \tau)\, v'_\mu(\mu)\, \mu'_\tau(T, \tau)]\, [w'_T(T, \tau)\, v(\mu) + v'_\mu(\mu)\, \mu'_T(T, \tau)] = \varphi(T)/L\}_{T=T_p}. \tag{14}$$

Let us present an example for utilizing the invariant solutions of [1] for the construction of solutions of boundary-value problems of heat conduction. Thus, an invariant solution for arbitrary f(T) having the form

$$x = \tau + \int [f(T)\, dT/(C - T)] = \tau + R(T), \tag{15}$$

has been established in [1], where $R(T) = x_0(T)$. Let the initial distribution $x_0(T)$ be given for $\tau = 0$:

$$x_0(T) = \int [f(T)\, dT/(C - T)]. \tag{16}$$

Then after differentiating (16) with respect to T we have

$$x'_0(T) = f(T)/(C - T). \tag{17}$$

If f(T) is known, then the function $x_0(T)$ should be selected so that the relation (17) is satisfied. On the other hand, if $x_0(T)$ is given, then the function f(T) is determined by (17).

The boundary conditions for the solution (15) can be formulated as follows.

A. The boundary moves according to the law $x = a(\tau)$ and the function $\psi(\tau)$ is defined on it. Then the relationship

$$a(\tau) = \tau + R(\psi(\tau)),$$

should be satisfied, from which the condition on $\psi(\tau)$ follows.

B. The boundary moves according to the law $x = a(\tau)$ and the function $x_T' = a(T)g(\tau)$ is defined on it so that by relying on the solution (15) we have the following system of equations

$$a(\tau) = \tau + R(T),$$
$$\varphi(T)\, g(\tau) = R'(T),$$

from which the condition on $g(\tau)$ (or on $a(\tau)$) follows in the form

$$a'(\tau) = 1 + g(\tau)\, \varphi(T).$$

C. Under the condition $x \to \infty$ as $T \to T_0$ a solution should exist on a half-line such that

$$R(T)|_{T\to T_0} \to \infty$$

(this is possible for $C = T_0$ and the function $f(T)$ bounded at the point $T = T_0$).

D. When we have $x \to \infty$ as $x_T' \to \infty$ on a half-line, then a solution should exist such that

$$x_T' = R'(T)|_{T\to T_0} \to \infty,$$

$$[x = \tau + R(T)]|_{T\to T_0} \to \infty.$$

This is also possible for $C = T_0$ and the function $f(T)$ bounded for $T = T_0$.

The discussions presented for the invariant solution (15) can also be continued for any other boundary conditions.

Making the results obtained specific, we indicate in the form of an example that the temperature field in a plate with an arbitrary dependence of the thermophysical characteristics of the material on the temperature, on whose bounding surfaces time dependences on the temperature $\psi_1(\tau)$ and $\psi_2(\tau)$ are maintained, is determined by the solution (15) for a selection of the initial distribution $x_0(T)$ and the laws of boundary motion $x_1 = a(\tau)$, $x_2 = b(\tau)$ that satisfy (16) and the dependences

$$a(\tau) = \tau + R(\psi_1(\tau)),$$

$$b(\tau) = \tau + R(\psi_2(\tau)),$$

respectively, when we give a specific value for the arbitrary constant C in the structure of the function R(T).

For the single-phase Stafan problem, the joint examination of the invariant solution (15) with the condition (12) on the front of the phase transition results in a unique value of the constant mentioned since it becomes equal to

$$C = T_p + f(T_p)L/\varphi(T_p). \tag{18}$$

For the mentioned value of the constant C the temperature field in a plate on one of whose boundary surfaces a phase transition of the substance occurs at the temperature T_p while a time dependence of the temperature $\psi(\tau)$ is maintained on its opposite surface, is determined as before by the solution (15) upon subjecting $x_0(T)$ to the distribution (16) and the laws of boundary motion, respectively, to the equations

$$a(\tau) = s(\tau) = \tau + R(T_p),$$

$$b(\tau) = \tau + R(\psi(\tau)).$$

The construction of the invariant solutions of two of the most complex nonlinear boundary-value problems of nonstationary heat conduction in a domain with a moving boundary is thereby shown.

Consideration of the results presented indicates that the invariant solutions of the heat-conduction equation obtained in [1] by using group analysis correspond to the solutions of boundary-value problems of heat conduction for a definite selection of the boundary conditions. The construction of such solutions is especially facilitated upon treatment of the process in displacements of isothermal surfaces.

In conclusion, we note that the approach developed in this paper is applicable in full measure to problems of filtration [2], laser action on a substance [3], solid fuel combustion [4], etc.

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